Port-Hamiltonian Systems for Fluid & Structural mechanics: Part I.

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ANR Project HAMECMOCPSYS on "*Hamiltonian Methods for the Control of Multidomain Distributed Parameter Systems*"

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Toulouse, France.

a 1-d.o.f. oscillator

Original dynamics: $m\ddot{x} + kx = 0\,,$ with $(x_0, \dot{x}_0)$ initial data , (1)

usually rewritten in state-space form, as:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}\,, \quad \text{with} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \text{initial data}\,. \tag{2}$$

$\Longrightarrow$ Choose the **Hamiltonian formalism**!

- **Energy variables**: position $q := x$, momentum $p := m\,\dot{x}$,

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- **Co-energy variables**: $\partial_q H = k\,q$ force, $\partial_p H = \frac{1}{m}\,p := \dot{x}$ velocity,
- **Dynamical system**:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} k\,q \\ \frac{1}{m}\,p \end{bmatrix} = J\,\boldsymbol{grad}_X H(X)\,.$$

with skew-symmetric matrix $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, i.e. $J^T = -J$.

a 1-d.o.f. oscillator (2)

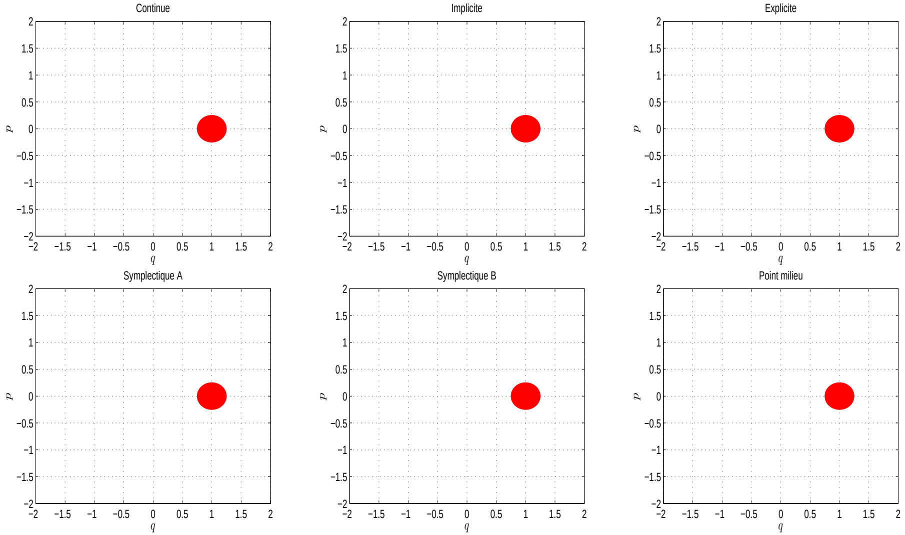
Theorem

J skew-symmetric $\Longrightarrow \frac{d}{dt}H(X(t)) = 0$.
Hence, the dynamical system is conservative, i.e. $H(X(t)) = H(X_0)$.

$$\begin{aligned}
\frac{dH}{dt}(t) &= (\boldsymbol{grad}_X H(X), \frac{d}{dt}X)_{\mathbb{R}^2} \\
&= (\boldsymbol{grad}_X H(X), J\,\boldsymbol{grad}_X H(X))_{\mathbb{R}^2} \\
&= 0\,, \quad \text{since } J \text{ is skew symmetric!}
\end{aligned}$$

■

Severe consequences on numerical simulation!



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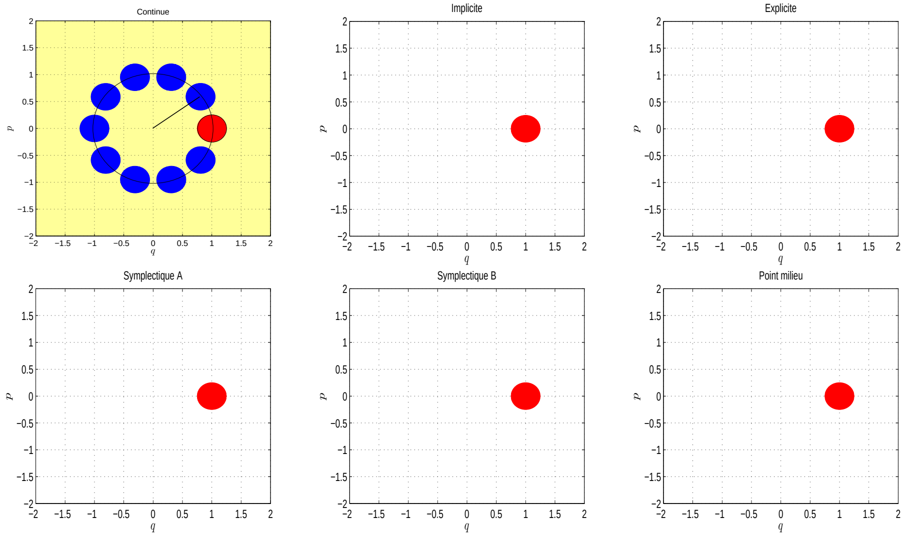

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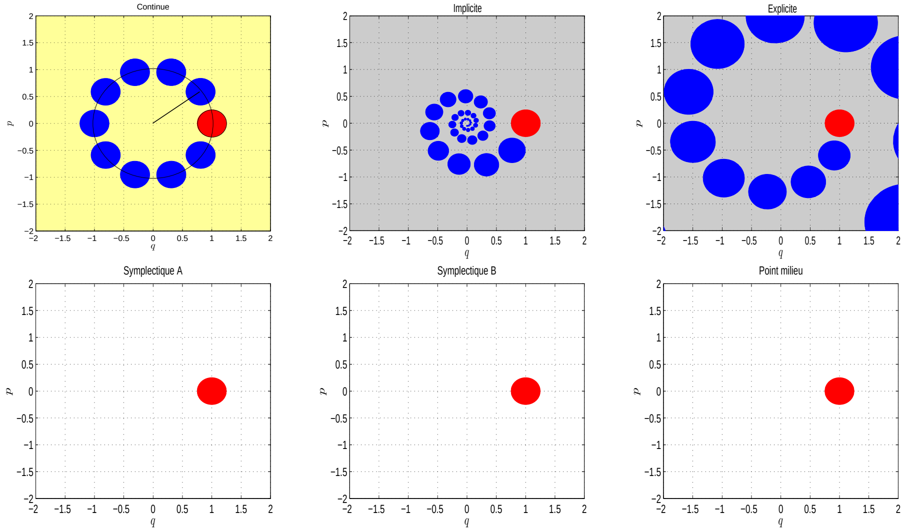

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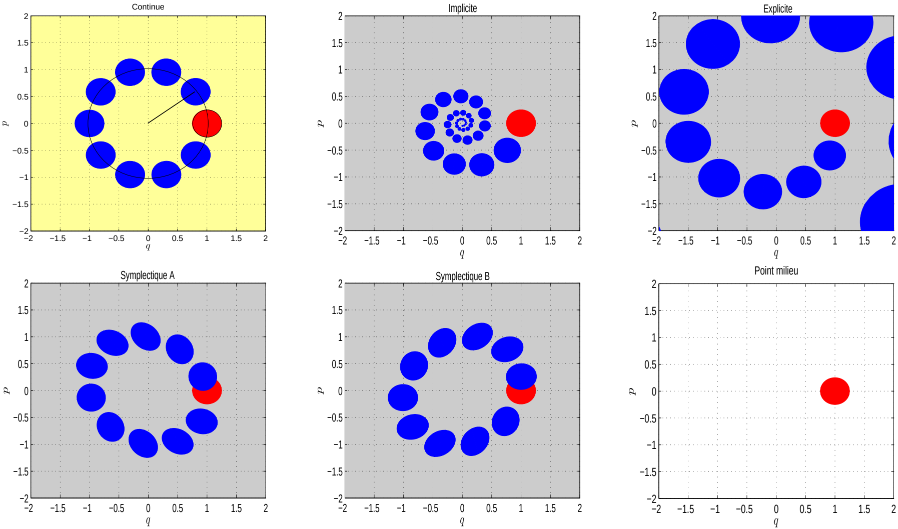

General ideas on Port Hamiltonian Systems (pHs)

1. strongly structured mathematical dynamical systems: both linear and non-linear, both finite-dimensional and infinite-dimensional,
2. based on physical grounds, allowing for different modelling levels,
3. all physics permitted: solid mechanics, structural mechanics, fluid mechanics, electromagnetism, electrical circuits, ...
4. comes along with specific numerical methods, which do preserve, at the discrete level, the structure of the continuous equations,[1]
5. allows for open dynamical systems, with interacting ports,
6. modularity: interconnection of sub-systems, and... easy multiphysics modelling, e.g. Fluid-Structure Interaction[2],
7. physically-based strategy for control and stabilization,
8. extensions to dissipative dynamical systems are available.

[1] come on June, 18th, to ROMA Seminar by Saïd Aoues!

[2] wait till... Part II, by Flávio Ribeiro, in a few minutes!

Outline

1. **Finite-dimensional case: Ordinary Differential Equations (ODEs)**
   - Useful Tools
   - Closed Systems
   - Open Systems
   - A short word on dissipation?
2. **Infinite-dimensional case: Partial Differential Equations (PDEs)**
   - New Useful Tools
   - Closed systems
   - More examples in fluid mechanics!
   - A short word on dissipation? Navier-Stokes, at last!
3. **Sloshing? a typical Fluid-Structure Interaction in Aeronautics!**

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Useful Tools in Finite Dimension

These 3 tools will be of major help in the study:

- the **gradient**: to be able to compute $\boldsymbol{grad}_X H(X)$, when $X \in \mathbb{R}^{2n}$,
- **skew symmetric** matrices: $J^T = -J$,
- in the special case of quadratic Hamiltonian, hence linear dynamical systems $\dot{X} = A\,X$: **matrix exponentials.**

  Straightforward consequences on **stability**:

  1. $\mathrm{spec}(A) \in \mathbb{C}_0^- \Longrightarrow (\exp(t\,A) \to 0$, as $t \to \infty)$, i.e. asymptotic stability.
  2. $\exists\lambda \in \mathrm{spec}(A)$, $\Re e(\lambda) > 0 \Longrightarrow (\|\exp(t\,A)\| \to \infty$, as $t \to \infty)$, i.e. (exponential) instability.
  3. the case with eigenvalues on $i\mathbb{R}$ is more subtle, and requires geometric insight on the eigenspaces to be solved: either stability or (polynomial) instability can be found.

Ex 1: the n-d.o.f. linear oscillator

Original dynamics: $M\ddot{x} + Kx = 0\,,$ with $(x_0, \dot{x}_0)$ initial data , (3)

with mechanical parameters $M = M^T > 0$, $K = K^T \geq 0$.

- Energy variables: $q := x \in \mathbb{R}^n$, $p := M\,\dot{x} \in \mathbb{R}^n$, set $X = (q,\, p) \in \mathbb{R}^{2n}$,
- Hamiltonian function: $H(X) := \frac{1}{2} p^T\, M^{-1}\, p + \frac{1}{2} q^T\, K\, q$,
- Co-energy variables: $\boldsymbol{grad}_q H = K\, q$, and $\boldsymbol{grad}_p H = M^{-1}\, p := \dot{x}$,
- Dynamical system in standard form:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} K\,q \\ M^{-1}\,p \end{bmatrix} = J\,\boldsymbol{grad}_X H(X)\,,$$

with $J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is skew-symmetric in $\mathbb{R}^{2n}$.

Ex.2: a 1-d.o.f. nonlinear oscillator: the pendulum

Original dynamics: $J\,\ddot{\theta} + g\,\sin(\theta) = 0$, with $(\theta_0, \dot{\theta}_0)$ initial data. (4)

$\Longrightarrow$ Choose the **Hamiltonian formalism**!

- **Energy variables**: position $q := \theta$, momentum $p := J\,\dot{\theta}$,
- **Hamiltonian function**: $H(X) := \frac{1}{2J}p^2 + g\,(1 - \cos(\theta))$,
- **Co-energy variables**: $\partial_q H = g\,\sin(\theta)$ torque, $\partial_p H = \frac{1}{J}p := \dot{\theta}$ angular velocity,
- **Dynamical system** with $X = (q,\,p)$:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} g\,\sin(\theta) \\ \dot{\theta} \end{bmatrix} = J\,\boldsymbol{grad}_X H(X)\,.$$

Theorem

J skew-symmetric $\Longrightarrow \frac{d}{dt}H(X(t)) = 0$. Hence, the **non-linear** *system is conservative, i.e. $H(X(t)) = H(X_0)$, with* **non-quadratic** *Hamiltonian.*

Open Systems, Ports, and Energy Balance (1)

Suppose we have **interaction** with the environment, by means of:

- actuators, with control $u \in \mathbb{R}^m$,
- sensors, with co-localized measurements or observations $y \in \mathbb{R}^m$,

then, the port-Hamiltonian system (pHs) is defined by:

$$\dot{X} = J\, \boldsymbol{grad}_X H(X) + g(X)\, u(t)\,, \quad (5)$$

$$y(t) = g(X)^T\, \boldsymbol{grad}_X H(X)\,. \quad (6)$$

Theorem

J skew-symmetric $\Longrightarrow$ the system is lossless. Indeed,
$\frac{d}{dt}H(X(t)) = (y(t), u(t))_{\mathbb{R}^m}$, *or* $H(X(t)) = H(X_0) + \int_0^t (y(\tau), u(\tau))_{\mathbb{R}^m}\, \mathrm{d}\tau$.

What about the linear / quadratic case?

Suppose the Hamiltonian function is quadratic $H(X) := \frac{1}{2}X^T\,Q\,X$, with $Q = Q^T > 0$, we then easily compute $\boldsymbol{grad}_X H(X) = QX$, and we can define the closed linear dynamical system:

$$\dot{X} = J\,Q\,X\,,$$

that is, the **matrix of the dynamics** reads $A := J\,Q$.

Let $B := g$ be the **control matrix** of size $n \times m$, then the open dynamical system is given by:

$$\dot{X} = J\,Q\,X + B\,u(t)\,, \qquad (7)$$

$$y(t) = B^T\,Q\,X\,. \qquad (8)$$

that is, the $m \times n$ **observation matrix** reads $C := B^T\,Q$.

Ex 1: n-d.o.f. damped oscillator

Original dynamics: $M\ddot{x}+(C+G)\dot{x}+Kx=0$, with $(x_0,\dot{x}_0)$ initial data, (9)

with $M=M^T>0$, $K=K^T\geq 0$ and $C=C^T$, $G=-G^T$.

- Energy variables: $q:=x\in\mathbb{R}^n$, $p:=M\,\dot{x}\in\mathbb{R}^n$,
- Hamiltonian function: $H(X)=\frac{1}{2}p^T\,M^{-1}\,p+\frac{1}{2}q^T\,K\,q$,
- Dynamical system in standard form:

$$\frac{d}{dt}X=\begin{bmatrix}0 & I\\ -I & -(G+C)\end{bmatrix}\,\boldsymbol{grad}_X H(X)=(J-R)\,\boldsymbol{grad}_X H(X)\,,$$

with $J:=\begin{bmatrix}0 & I\\ -I & -G\end{bmatrix}$, and $R:=\begin{bmatrix}0 & 0\\ 0 & C\end{bmatrix}$.

$\Longrightarrow$ Examine the two terms separately :

- Role of the skew-symmetric G matrix, the gyroscopic term.
- Damping effect is ensured, provided that $C=C^T\geq 0$.

Ex 1: about the G matrix

1. This matrix is often not considered in modelling processes of damping! Why?
   When $C = 0$, whatever $G = -G^T$, the system proves conservative: $\frac{d}{dt} H_0(X(t)) = 0$!
2. Is it a naive generalizations due to mathematicians? No!
   Classical mechanical example: Coriolis force $f = \omega \wedge \dot{x}$, with rotational speed $\omega = (p, q, r)^T$, $f = G_\omega \dot{x}$ where
   $G_\omega := \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$ is skew symmetric.
3. Also in electromagnetics: Lorentz force in a uniform magnetic field.

Open Systems with Damping, Energy Balance (2)

The port-Hamiltonian system (pHs) is defined by:

$$\dot{X} = (J - R)\, \boldsymbol{grad}_X H(X) + g(X)\, u(t)\,, \tag{10}$$

$$y(t) = g(X)^T\, \boldsymbol{grad}_X H(X)\,. \tag{11}$$

Theorem

J skew-symmetric and R positive $\Longrightarrow$ the system is passive. Indeed,

$$\begin{aligned} \frac{d}{dt} H(X(t)) &= -(\boldsymbol{grad}_X H(X), R\, \boldsymbol{grad}_X H(X))_{\mathbb{R}^n} + (y(t), u(t))_{\mathbb{R}^m}\,, \\ &\leq (y(t), u(t))_{\mathbb{R}^m}\,. \end{aligned}$$

Hence, $H(X(t)) - H(X_0) \leq \int_0^t (y(\tau), u(\tau))_{\mathbb{R}^m}\, \mathrm{d}\tau$.

Outline

Useful Notions in Infinite Dimension

These notions will be of major help in the following:

- **functions** u, instead of vectors X,
- an infinite-dimensional **Hilbert functional space** $\mathcal{H}$ for functions, instead of a finite-dimensional Euclian vector space $\mathbb{R}^{2n}$,
- a Hamiltonian **functional** H, defined on functions u, instead of a Hamiltonian function defined on vectors, e.g.:

$$\begin{array}{rcl} H:\mathcal{H} & \to & \mathbb{R} \\ u & \mapsto & H(u) := \dfrac{1}{2}\displaystyle\int_0^L u(z)^2\,\mathrm{d}z \end{array}$$

Useful Tools in Infinite Dimension

These tools will be of major help in the following:

- the **variational derivative** of a functional: $\delta_X H$, in place of the gradient of the function, defined by

$$H(u + \varepsilon\, w) = H(u) + \varepsilon\,(\delta_u H,\, w)_{\mathcal{H}} + O(\varepsilon^2)$$

  N.B. in the above easy example, $\delta_u H = u$.

- formally **skew symmetric operators**: $\mathcal{J}^T = -\mathcal{J}$, w.r.t the scalar product in the Hilbert space $\mathcal{H}$, i.e.

$$(u,\, \mathcal{J} v)_{\mathcal{H}} = -(\mathcal{J} u,\, v)_{\mathcal{H}}$$

- in the special case of quadratic Hamiltonian, hence linear dynamical systems $\dot{X} = \mathcal{A} X$: **semigroups**[3].

[3] come on June, 4th, to ROMA Seminar by Ghislain Haine!

Ex 3: Webster horn equations (1)

• the energy variables are: density ρ, and particle velocity v,
• with pressure $p := c_0^2\,\rho$, and energy density $U(\rho) := \frac{c_0^2}{2\,\rho_0^2}\rho^2$, let us define the Hamiltonian:

$$H(\rho, v) := \int_0^L \left(\frac{1}{2}\rho_0 v^2 + \rho_0 U(\rho)\right) S(z)\,\mathrm{d}z,$$

• the co-energy variables are: $\delta_\rho H = \rho_0\,U'(\rho) = \frac{1}{\rho_0}\,p$ and $\delta_v H = \rho_0\,v$, • then, the corresponding port-Hamiltonian system is given by

$$\frac{d}{dt}\begin{bmatrix}\rho \\ v\end{bmatrix} = \begin{bmatrix}0 & -\frac{1}{S}\partial_z\left(S\;\cdot\;\right) \\ -\partial_z & 0\end{bmatrix}\begin{bmatrix}\delta_\rho H \\ \delta_v H\end{bmatrix}. \qquad (12)$$

• The wave equations which govern the acoustic pressure p and the particle velocity v are given by, respectively:

$$\frac{1}{c_0^2}\,\partial_t^2 p(z,t) - \frac{1}{S(z)}\,\partial_z\left[S(z)\partial_z p(z,t)\right] = 0, \qquad (13)$$

$$\frac{1}{c_0^2}\,\partial_t^2 v(z,t) - \partial_z\left[\frac{1}{S(z)}\,\partial_z\left[S(z)v(z,t)\right]\right] = 0, \qquad (14)$$

Ex 4: compressible Euler equations

Consider an inviscid, irrotational and isentropic fluid, in $\Omega \subset \mathbb{R}^3$:

$$\frac{d}{dt}\rho = -div(\rho\, \boldsymbol{v}) \quad (15)$$

$$\frac{d}{dt}\boldsymbol{v} = -(\boldsymbol{v}.\boldsymbol{grad})\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{grad} p\,. \quad (16)$$

Following e.g. [van der Schaft & Maschke, 2001],

- Energy variables: ρ, $\boldsymbol{v}$,
- Hamiltonian: $H(\rho, \boldsymbol{v}) := \int_\Omega \left(\frac{1}{2}\rho \boldsymbol{v}.\boldsymbol{v} + \rho U(\rho)\right)\, \mathrm{d}V$,
- Co-energy variables: $\delta_\rho H = \frac{1}{2}\boldsymbol{v}.\boldsymbol{v} + h(\rho)$ and $\delta_{\boldsymbol{v}} H = \rho\, \boldsymbol{v}$,
- Dynamical system in standard form:

$$\frac{d}{dt}\begin{bmatrix}\rho \\ \boldsymbol{v}\end{bmatrix} = \begin{bmatrix} 0 & -div \\ -\boldsymbol{grad} & 0 \end{bmatrix} \begin{bmatrix}\delta_\rho H \\ \delta_{\boldsymbol{v}} H\end{bmatrix} ;$$

Definitions for pHs (1)

Consider the dynamical system

$$\frac{d}{dt}X(z,t) = (\mathcal{J})\ \delta_X \mathcal{H}(X) \tag{17}$$

with (quadratic) Hamiltonian :

$$\mathcal{H}(X) = \frac{1}{2}\int_0^L X(z,t)^T \mathcal{L} X(z,t)\,\mathrm{d}z\,,$$

and (linear) variational derivative :

$$\delta_X \mathcal{H}(X) = \mathcal{L} X(z,t)\,.$$

We suppose that operator $\mathcal{J}$ is formally skew-symmetric.

Definitions for pHs (2)

The energy balance associated to this system is:

$$\begin{aligned}
\frac{d\mathcal{H}}{dt}(t) &= \int_0^L \delta_X \mathcal{H}(X).\frac{dX}{dt}\,\mathrm{d}z\,,\\
&= \int_0^L \delta_X \mathcal{H}(X).(\mathcal{J}).\delta_X \mathcal{H}(X)\,\mathrm{d}z\,,\\
&= 0\,, \quad \text{since } \mathcal{J} \text{ is skew-adjoint?} \quad \text{Almost!}
\end{aligned}$$

For the horn equation, say with uniform cross-section $S(z) = S$,

$$\begin{aligned}
\frac{d\mathcal{H}}{dt}(t) &= \int_0^L (-e_1\,\partial_z e_2 - e_2\,\partial_z e_1)\,\mathrm{d}z\,,\\
&= \int_0^L -\partial_z(e_1\,e_2)\,\mathrm{d}z\,,\\
&= e_1(0)\,e_2(0) - e_1(L)\,e_2(L)\,.
\end{aligned}$$

$\Longrightarrow$ Energy flows through the boundary, only!

More involved examples in fluid mechanics

Some more specific Hamiltonian models for fluid mechanics are available for:

- potential flow (irrotational fluid), with the use of the **stream function**
- incompressible fluid (very difficult: an infinite-dimensional Differential Algebraic Equation!), with the constraint $div(\mathbf{v}) = 0$, make use of the **vorticity vector** $\boldsymbol{\omega} := \boldsymbol{curl}(\mathbf{v})$.

$\Longrightarrow$ read the book by [Olver, 1993]

$\Longrightarrow$ come and talk with me, since I have read the book for you!

Definitions for pHs (1)

Consider the dynamical system

$$\frac{d}{dt}X(z,t) = (\mathcal{J} - \mathcal{R})\ \delta_X \mathcal{H}_0(X) \tag{18}$$

with (quadratic) Hamiltonian :

$$\mathcal{H}_0(X) = \frac{1}{2}\int_0^L X(z,t)^T \mathcal{L} X(z,t)\,\mathrm{d}z\,,$$

and (linear) variational derivative :

$$\delta_X \mathcal{H}_0(X) = \mathcal{L}X(z,t)\,.$$

We suppose that

- operator $\mathcal{J}$ is formally skew-symmetric,
- operator $\mathcal{R}$ is positive self-adjoint.

Definitions for pHs (2)

The energy balance associated to this system is:

$$\begin{aligned}
\frac{d\mathcal{H}_0}{dt}(t) &= \int_0^L \delta_X \mathcal{H}_0(X).\frac{dX}{dt}\,\mathrm{d}z\,, \\
&= \int_0^L \delta_X \mathcal{H}_0(X).(\mathcal{J}-\mathcal{R}).\delta_X \mathcal{H}_0(X)\,\mathrm{d}z\,, \\
&= -\int_0^L \delta_X \mathcal{H}_0(X).\mathcal{R}.\delta_X \mathcal{H}_0(X)\,\mathrm{d}z\,, \\
&\leq 0\,.
\end{aligned}$$

But when $\mathcal{R} \neq 0$, no underlying Dirac structure is to be found for this damped system, with efforts $e := \delta_X \mathcal{H}_0(X)$ and flows $f := \frac{dX}{dt}$, linked by $f = \mathcal{J}\, e$.

Ex 5: compressible Navier-Stokes

Consider an irrotational and isentropic fluid, in $\Omega \subset \mathbb{R}^3$:

$$\frac{d}{dt}\rho = -div(\rho \boldsymbol{v}) \quad (19)$$

$$\frac{d}{dt}\boldsymbol{v} = -(\boldsymbol{v}.\boldsymbol{grad})\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{grad}p + \frac{1}{Re}\boldsymbol{\Delta v}\,. \quad (20)$$

Still following [van der Schaft & Maschke, 2001],

- Energy variables: ρ, $\boldsymbol{v}$,
- Hamiltonian: $H := \int_\Omega \left(\frac{1}{2}\rho\boldsymbol{v}.\boldsymbol{v} + \rho U(\rho)\right)\,\mathrm{d}V$,
- Co-energy variables: $\delta_\rho H = \frac{1}{2}\boldsymbol{v}.\boldsymbol{v} + h(\rho)$ and $\delta_{\boldsymbol{v}} H = \rho\,\boldsymbol{v}$,
- Dynamical system in standard form:

$$\frac{d}{dt}\begin{bmatrix}\rho \\ \boldsymbol{v}\end{bmatrix} = \begin{bmatrix}0 & -div \\ -\boldsymbol{grad} & 0\end{bmatrix}\begin{bmatrix}\delta_\rho H_0 \\ \delta_{\boldsymbol{v}} H_0\end{bmatrix} - \begin{bmatrix}0 & 0 \\ 0 & \mathcal{C}\end{bmatrix}\begin{bmatrix}\delta_\rho H \\ \delta_{\boldsymbol{v}} H\end{bmatrix}\;;$$

Ex 5: compressible Navier-Stokes

With $\mathcal{C} = -\frac{1}{Re}\boldsymbol{\Delta}$. It has the desired $(\mathcal{J} - \mathcal{R})$ form:

- $\mathcal{J}$ is a skew-symmetric operator, since the formal adjoint of div is $-\boldsymbol{grad}$,
- $\mathcal{R}$ is a symmetric and positive operator, since $-\boldsymbol{\Delta}$ is.

$\Longrightarrow$ more important, the parametrization $\mathcal{R} = \mathcal{G}\mathcal{S}\mathcal{G}^*$ is very easily found to be:

$$\mathcal{G} := \begin{bmatrix} 0 \\ \boldsymbol{grad} \end{bmatrix}, \quad \mathcal{G}^* = \begin{bmatrix} 0 & -div \end{bmatrix}, \quad \text{and} \quad \mathcal{S} := \frac{1}{Re} I \,.$$

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3. **Sloshing? a typical Fluid-Structure Interaction in Aeronautics!**

and now? the show must go on!

Let us welcome: Flávio Ribeiro!